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The onset of convection in a bidisperse porous medium

D.A. Nield [a], A.V. Kuznetsov [b,*]

[a] Department of Engineering Science, University of Auckland, Private Bag 92019, Auckland, New Zealand
[b] Department of Mechanical and Aerospace Engineering, North Carolina State University, Campus Box 7910, Raleigh, NC 27695-7910, United States

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Abstract

The classical Rayleigh–Bénard theory, for the onset of convection in a horizontal layer uniformly heated from below, has been applied to a bidisperse porous medium. The linear stability analysis leads to an expression for the critical Rayleigh number as a function of a Darcy number, two volume fractions, a permeability ratio, a thermal capacity ratio, a thermal conductivity ratio, an inter-phase heat transfer parameter and an inter-phase momentum transfer parameter.

1. Introduction

A bidisperse porous medium (BDPM, see Fig. 1), as informally defined by Chen et al. [1,2], is composed of clusters of large particles that are agglomerations of small particles. Thus there are macro-pores between the clusters and micro-pores within them. Applications are found in bidisperse adsorbent or bidisperse capillary wicks in a heat pipe. Since the bidisperse wick structure significantly increases the area available for liquid film evaporation, it has been proposed for use in the evaporator of heat pipes.

A BDPM may thus be looked at as a standard porous medium in which the solid phase is replaced by another porous medium, whose temperature may be denoted by T_p if local thermal equilibrium is assumed within each cluster. We can then talk about the f-phase (the macro-pores) and the p-phase (the remainder of the structure). An alternative way of looking at the structure is to regard it as a porous medium in which fractures or tunnels have been introduced. One can then think of the f-phase as being a 'fracture phase' and the p-phase as being a 'porous phase'.

Extending the Brinkman model for a monodisperse porous medium, Nield and Kuznetsov [3] modeled the steady-state momentum transfer in a BDPM by the following pair of coupled equations for $\mathbf{v}_f^*$ and $\mathbf{v}_p^*$, where the asterisks denote dimensional variables

$$\mathbf{G} = \left(\frac{\mu}{K_f}\right)\mathbf{v}_f^* + \zeta(\mathbf{v}_f^* - \mathbf{v}_p^*) - \tilde{\mu}_f \nabla^{*2}\mathbf{v}_f^*, \tag{1}$$

$$\mathbf{G} = \left(\frac{\mu}{K_p}\right)\mathbf{v}_p^* + \zeta(\mathbf{v}_p^* - \mathbf{v}_f^*) - \tilde{\mu}_p \nabla^{*2}\mathbf{v}_p^*. \tag{2}$$

Here $\mathbf{G}$ is the negative of the applied pressure gradient, μ is the fluid viscosity, $\tilde{\mu}_f$ and $\tilde{\mu}_p$ are the effective viscosities in the two phases, K_f and K_p are the permeabilities of the two phases, and ζ is the coefficient for momentum transfer between the two phases. From Eqs. (1) and (2), $\mathbf{v}_p^*$ can be eliminated to give

$$\tilde{\mu}_f\tilde{\mu}_p\nabla^{*4}\mathbf{v}_f^* - [\tilde{\mu}_f(\zeta + \mu/K_p) + \tilde{\mu}_p(\zeta + \mu/K_f)]\nabla^{*2}\mathbf{v}_f^* + [\zeta\mu(1/K_f + 1/K_p) + \mu^2/(K_fK_p)]\mathbf{v}_f^* = \mathbf{G}(2 + \mu/K_p) \tag{3}$$

and $\mathbf{v}_p^*$ is given by the same equation with subscripts swapped.

These equations were applied by Nield and Kuznetsov [4,5] to forced convection in a channel.

In this paper we apply a two-velocity two-temperature formulation to the Horton–Rogers–Lapwood problem, following the procedure used by Banu and Rees [6].

* Corresponding author. Tel.: +1 919 515 5292; fax: +1 919 515 7968.
E-mail address: avkuznet@eos.ncsu.edu (A.V. Kuznetsov).

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Nomenclature

c	specific heat at constant pressure
d	layer depth
Da_f	Darcy number, $\frac{\tilde{\mu}K_f}{\mu d^2}$
$\mathbf{G}$	negative of the applied pressure gradient
h	inter-phase heat transfer coefficient (incorporating the specific area)
H	inter-phase heat transfer parameter
k	thermal conductivity
K	permeability
m	dimensionless horizontal wavenumber of the disturbance
Ra_f	Rayleigh number, $\frac{\rho_F g\hat{\beta}(T_l-T_u)K_f d}{\mu\phi k_f/(\rho c)_f}$
Ra	$\gamma Ra_f/(\gamma+1)$
T_l	temperature at the lower boundary ($y^*=0$)
T_u	temperature at the upper boundary ($y^*=d$)
T_0	reference temperature, T_l-T_u
$\mathbf{v}^*$	filtration velocity

Greek symbols

α	thermal diffusivity ratio, $\frac{k_f}{k_p}\frac{(\rho c)_p}{(\rho c)_f}$
β	modified thermal capacity ratio, $\frac{(1-\phi)k_p(\rho c)_f}{\phi k_f(\rho c)_p}$
$\hat{\beta}$	volumetric thermal expansion coefficient of the fluid
γ	modified thermal conductivity ratio, $\frac{\phi k_f}{(1-\phi)k_p}$
ε	volume fraction of the p-phase
ζ	coefficient for momentum transfer between the two phases
μ	fluid viscosity
$\tilde{\mu}$	effective viscosity of the porous medium
ρ_F	density of the fluid
σ_f	f-phase momentum transfer parameter, $\frac{\zeta K_f}{\mu}$
ϕ	volume fraction of the f-phase

Subscripts

f	fracture phase (macro-pores)
p	porous phase (micro-pores)

Superscript

*	dimensional variable

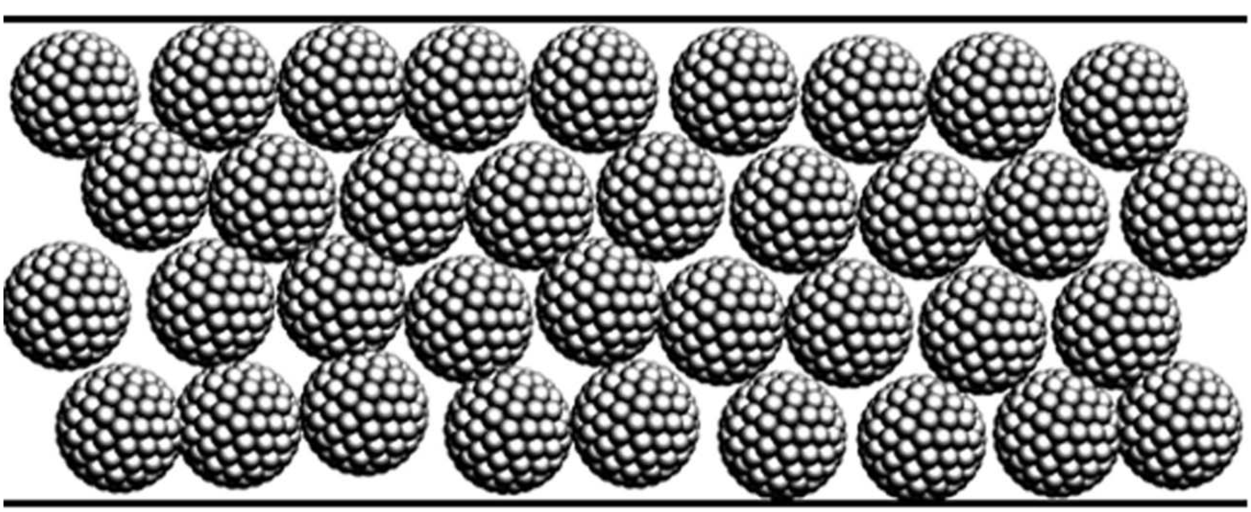

Fig. 1. Sketch of a bidisperse porous medium.

2. Analysis

We consider a layer of a BDPM heated uniformly from below, with applied temperatures T_l and T_u at the lower boundary ($y^*=0$) and the upper boundary ($y^*=d$), respectively. (The asterisks denote dimensional variables.) The equations of continuity (expressing conservation of mass) for the velocity components in the two phases are

$$\frac{\partial u_f^*}{\partial x^*}+\frac{\partial v_f^*}{\partial y^*}=0, \tag{4}$$

$$\frac{\partial u_p^*}{\partial x^*}+\frac{\partial v_p^*}{\partial y^*}=0. \tag{5}$$

We note that in the traditional Darcy formulation the pressure is an intrinsic quantity, i.e., it is the pressure in the fluid. We recognize that in a BDPM the fluid occupies all of the f-phase and a fraction of the p-phase. We denote the volume fraction of the f-phase by ϕ (something that in a regular porous medium would be called the porosity) and the porosity in the p-phase by ε. Thus $1-\phi$ is the volume fraction of the p-phase, and the volume fraction of the BDPM occupied by the fluid is $\phi+(1-\phi)\varepsilon$. The volume average of the temperature over the fluid is

$$T_{\mathrm{F}}^{*}=\frac{\phi T_{\mathrm{f}}^{*}+(1-\phi)\varepsilon T_{\mathrm{p}}^{*}}{\phi+(1-\phi)\varepsilon}. \tag{6}$$

The drag force (per unit volume) balances the gradient of the excess pressure over hydrostatic. Our basic hypothesis is that in a BDPM the drag is increased by an amount $\zeta(\mathbf{v}_{\mathrm{f}}^{*}-\mathbf{v}_{\mathrm{p}}^{*})$ for the f-phase and decreased by the same amount for the p-phase. Accordingly, we write the momentum equations as

$$\frac{\partial p^{*}}{\partial x^{*}}=-\frac{\mu}{K_{\mathrm{f}}}u_{\mathrm{f}}^{*}-\zeta(u_{\mathrm{f}}^{*}-u_{\mathrm{p}}^{*})+\tilde{\mu}\nabla^{2}u_{\mathrm{f}}^{*}, \tag{7}$$

$$\frac{\partial p^{*}}{\partial x^{*}}=-\frac{\mu}{K_{\mathrm{p}}}u_{\mathrm{p}}^{*}-\zeta(u_{\mathrm{p}}^{*}-u_{\mathrm{f}}^{*})+\tilde{\mu}\nabla^{2}u_{\mathrm{p}}^{*}, \tag{8}$$

$$\frac{\partial p^{*}}{\partial y^{*}}=-\frac{\mu}{K_{\mathrm{f}}}v_{\mathrm{f}}^{*}-\zeta(v_{f}^{*}-v_{\mathrm{p}}^{*})+\tilde{\mu}\nabla^{2}v_{\mathrm{f}}^{*}+\rho_{\mathrm{F}}g\hat{\beta}(T_{\mathrm{F}}^{*}-T_{0}), \tag{9}$$

$$\frac{\partial p^{*}}{\partial y^{*}}=-\frac{\mu}{K_{\mathrm{p}}}v_{\mathrm{p}}^{*}-\zeta(v_{\mathrm{p}}^{*}-v_{\mathrm{f}}^{*})+\tilde{\mu}\nabla^{2}v_{\mathrm{p}}^{*}+\rho_{\mathrm{F}}g\hat{\beta}(T_{\mathrm{F}}^{*}-T_{0}). \tag{10}$$

We have simplified the equations by assuming that $\tilde{\mu}_{\mathrm{f}}$ and $\tilde{\mu}_{\mathrm{p}}$ are equal, so the subscripts on $\tilde{\mu}$ can be dropped. Here ρ_{F} is the density of the fluid, $\hat{\beta}$ is the volumetric thermal expansion coefficient of the fluid, and T_0 is a reference temperature.

The thermal energy equations are taken as

$$\phi(\rho c)_{\mathrm{f}}\frac{\partial T_{\mathrm{f}}^{*}}{\partial t^{*}}+\phi(\rho c)_{\mathrm{f}}\mathbf{v}_{\mathrm{f}}^{*}\cdot\nabla T_{\mathrm{f}}^{*}=\phi k_{\mathrm{f}}\nabla^{2}T_{\mathrm{f}}^{*}+h(T_{\mathrm{p}}^{*}-T_{\mathrm{f}}^{*}), \tag{11}$$

$$(1-\phi)(\rho c)_{\mathrm{p}}\frac{\partial T_{\mathrm{p}}^{*}}{\partial t^{*}}+(1-\phi)(\rho c)_{\mathrm{p}}\mathbf{v}_{\mathrm{p}}^{*}\cdot\nabla T_{\mathrm{p}}^{*}=(1-\phi)k_{\mathrm{p}}\nabla^{2}T_{\mathrm{p}}^{*}+h(T_{\mathrm{f}}^{*}-T_{\mathrm{p}}^{*}). \tag{12}$$

Here c denotes the specific heat at constant pressure, k denotes the thermal conductivity, and h is an inter-phase heat transfer coefficient (incorporating the specific area).

We introduce dimensionless variables as follows:

$$(x^{*},y^{*})=d(x,y),\quad t^{*}=\frac{(\rho c)_{\mathrm{f}}}{k_{\mathrm{f}}}d^{2}t,\quad p^{*}=\frac{k_{\mathrm{f}}\mu}{(\rho c)_{\mathrm{f}}K_{\mathrm{f}}}p, \tag{13}$$

$$(u_{\mathrm{f}}^{*},v_{\mathrm{f}}^{*})=\frac{\phi k_{\mathrm{f}}}{(\rho c)_{\mathrm{f}}d}(u_{\mathrm{f}},v_{\mathrm{f}}),\quad (u_{\mathrm{p}}^{*},v_{\mathrm{p}}^{*})=\frac{(1-\phi)k_{\mathrm{p}}}{(\rho c)_{\mathrm{p}}d}(u_{\mathrm{p}},v_{\mathrm{p}}), \tag{14}$$

$$T_{\mathrm{f}}^{*}=(T_{\mathrm{l}}-T_{\mathrm{u}})\theta_{\mathrm{f}}+T_{\mathrm{u}},\quad T_{\mathrm{p}}^{*}=(T_{\mathrm{l}}-T_{\mathrm{u}})\theta_{\mathrm{p}}+T_{\mathrm{u}}. \tag{15}$$

We take the reference temperature T_0 as $T_{\mathrm{l}}-T_{\mathrm{u}}$. We also introduce the stream functions ψ_{f} and ψ_{p} defined so that

$$u_{\mathrm{f}}=-\frac{\partial\psi_{\mathrm{f}}}{\partial y},\quad v_{\mathrm{f}}=\frac{\partial\psi_{\mathrm{f}}}{\partial x},\quad u_{\mathrm{p}}=-\frac{\partial\psi_{\mathrm{p}}}{\partial y},\quad v_{\mathrm{p}}=\frac{\partial\psi_{\mathrm{p}}}{\partial x}. \tag{16}$$

We define a Rayleigh number Ra_{f} and a Darcy number Da_{f} based on properties in the f-phase by

$$Ra_{\mathrm{f}}=\frac{\rho_{\mathrm{F}}g\hat{\beta}(T_{\mathrm{l}}-T_{\mathrm{u}})K_{\mathrm{f}}d}{\mu\phi k_{\mathrm{f}}/(\rho c)_{\mathrm{f}}},\quad Da_{\mathrm{f}}=\frac{\tilde{\mu}K_{\mathrm{f}}}{\mu d^{2}}. \tag{17a, b}$$

Elimination of the pressure from Eqs. (7)–(10) gives

$$\left[(1+\sigma_{\mathrm{f}})\nabla^{2}-Da_{\mathrm{f}}\nabla^{4}\right]\psi_{\mathrm{f}}-\beta\sigma_{\mathrm{f}}\nabla^{2}\psi_{\mathrm{p}}=Ra_{\mathrm{f}}\frac{\partial\theta_{\mathrm{F}}}{\partial x}, \tag{18}$$

$$-\sigma_{\mathrm{f}}\nabla^{2}\psi_{\mathrm{f}}+\beta\left[\left(\frac{1}{K_{\mathrm{r}}}+\sigma_{\mathrm{f}}\right)\nabla^{2}-Da_{\mathrm{f}}\nabla^{4}\right]\psi_{\mathrm{p}}=Ra_{\mathrm{f}}\frac{\partial\theta_{\mathrm{F}}}{\partial x}, \tag{19}$$

where

$$\frac{\partial\theta_{\mathrm{F}}}{\partial x}=\frac{\phi\frac{\partial\theta_{\mathrm{f}}}{\partial x}+(1-\phi)\varepsilon\frac{\partial\theta_{\mathrm{p}}}{\partial x}}{\phi+(1-\phi)\varepsilon}. \tag{20}$$

Here we have introduced the dimensionless parameters

$$\sigma_{\mathrm{f}}=\frac{\zeta K_{\mathrm{f}}}{\mu},\quad \beta=\frac{(1-\phi)k_{\mathrm{p}}(\rho c)_{\mathrm{f}}}{\phi k_{\mathrm{f}}(\rho c)_{\mathrm{p}}}. \tag{21}$$

Thus σ_{f} is an inter-phase momentum transfer parameter, while β is a modified thermal diffusivity ratio.

Also, the thermal energy Eqs. (11) and (12) become

$$\frac{\partial\theta_{\mathrm{f}}}{\partial t}-\frac{\partial\psi_{\mathrm{f}}}{\partial y}\frac{\partial\theta_{\mathrm{f}}}{\partial x}+\frac{\partial\psi_{\mathrm{f}}}{\partial x}\frac{\partial\theta_{\mathrm{f}}}{\partial y}=\nabla^{2}\theta_{\mathrm{f}}+H(\theta_{\mathrm{p}}-\theta_{\mathrm{f}}), \tag{22}$$

$$\alpha\frac{\partial\theta_{\mathrm{p}}}{\partial t}-\frac{\partial\psi_{\mathrm{p}}}{\partial y}\frac{\partial\theta_{\mathrm{p}}}{\partial x}+\frac{\partial\psi_{\mathrm{p}}}{\partial x}\frac{\partial\theta_{\mathrm{p}}}{\partial y}=\nabla^{2}\theta_{\mathrm{p}}+\gamma H(\theta_{\mathrm{f}}-\theta_{\mathrm{p}}), \tag{23}$$

where

$$\alpha=\frac{k_{\mathrm{f}}}{k_{\mathrm{p}}}\frac{(\rho c)_{\mathrm{p}}}{(\rho c)_{\mathrm{f}}},\quad \gamma=\frac{\phi k_{\mathrm{f}}}{(1-\phi)k_{\mathrm{p}}},\quad H=\frac{hd^{2}}{\phi k_{\mathrm{f}}}. \tag{24}$$

Thus α is a thermal diffusivity ratio, γ is a modified thermal conductivity ratio, and H is an inter-phase heat transfer parameter.

The conducting state solution is

$$\psi_{\mathrm{f}}=\psi_{\mathrm{p}}=0,\quad \theta_{\mathrm{f}}=\theta_{\mathrm{p}}=1-y. \tag{25}$$

We now perturb this solution and write

$$\psi_{\mathrm{f}}=\Psi_{\mathrm{f}},\quad \psi_{\mathrm{p}}=\Psi_{\mathrm{p}},\quad \theta_{\mathrm{f}}=1-y+\Theta_{\mathrm{f}},\quad \theta_{\mathrm{p}}=1-y+\Theta_{\mathrm{p}}. \tag{26}$$

We also invoke the principle of exchange of stabilities. This has the affect that the inertial coefficient α drops out of the subsequent equations. Substitution in Eqs. (18)–(23) and linearization gives

$$[(1+\sigma_{\mathrm{f}})\nabla^{2}-Da_{\mathrm{f}}\nabla^{4}]\Psi_{\mathrm{f}}-\beta\sigma_{\mathrm{f}}\nabla^{2}\Psi_{\mathrm{p}}=Ra_{\mathrm{f}}\left[\frac{\phi\frac{\partial\Theta_{\mathrm{f}}}{\partial x}+(1-\phi)\varepsilon\frac{\partial\Theta_{\mathrm{p}}}{\partial x}}{\phi+(1-\phi)\varepsilon}\right], \tag{27}$$

$$-\sigma_{\mathrm{f}}\nabla^{2}\Psi_{\mathrm{f}}+\beta\left[\left(\frac{1}{K_{\mathrm{r}}}+\sigma_{\mathrm{f}}\right)\nabla^{2}-Da_{\mathrm{f}}\nabla^{4}\right]\Psi_{\mathrm{p}}=Ra_{\mathrm{f}}\left[\frac{\phi\frac{\partial\Theta_{\mathrm{f}}}{\partial x}+(1-\phi)\varepsilon\frac{\partial\Theta_{\mathrm{p}}}{\partial x}}{\phi+(1-\phi)\varepsilon}\right], \tag{28}$$

$$\frac{\partial \Theta_f}{\partial t} = \nabla^2 \Theta_f + \frac{\partial \Psi_f}{\partial x} + H(\Theta_p - \Theta_f), \tag{29}$$

$$\alpha \frac{\partial \Theta_p}{\partial t} = \nabla^2 \Theta_p + \frac{\partial \Psi_p}{\partial x} + \gamma H(\Theta_f - \Theta_p). \tag{30}$$

For perfectly conducting and stress-free boundaries the boundary conditions are

$$\Psi_f = \Psi_p = \frac{\partial^2 \Psi_f}{\partial y^2} = \frac{\partial^2 \Psi_p}{\partial y^2} = \Theta_f = \Theta_p = 0 \quad \text{at} \quad y = 0 \quad \text{and at} \quad y = 1. \tag{31}$$

The solution of the system of Eqs. (27)–(31) is

$$\Psi_f = A_1 \sin \pi y \cos mx, \quad \Psi_p = A_2 \sin \pi y \cos mx,$$
$$\Theta_f = A_3 \sin \pi y \sin mx, \quad \Theta_p = A_4 \sin \pi y \sin mx, \tag{32a, b, c, d}$$

where $A_1, \ldots, A_4$ are constants and m denotes the horizontal wavenumber of the disturbance.

Substitution of Eq. (32) into Eqs. (27)–(30), and elimination of $A_1, \ldots, A_4$, yields the eigenvalue equation

$$\begin{vmatrix} (1+\sigma_f)M + Da_f M^2 & -\beta\sigma_f M & \frac{m\phi Ra_f}{\phi+(1-\phi)\varepsilon} & \frac{m(1-\phi)\varepsilon Ra_f}{\phi+(1-\phi)\varepsilon} \\ -\sigma_f M & \beta\left[\left(\frac{1}{K_r}+\sigma_f\right)M + Da_f M^2\right] & \frac{m\phi Ra_f}{\phi+(1-\phi)\varepsilon} & \frac{m(1-\phi)\varepsilon Ra_f}{\phi+(1-\phi)\varepsilon} \\ m & 0 & M+H & -H \\ 0 & m & -\gamma H & M+\gamma H \end{vmatrix} = 0 \tag{33}$$

Here M is shorthand for $\pi^2 + m^2$.

Expanding and solving for Ra_f gives

$$Ra_f = \frac{\beta[\phi + (1-\phi)\varepsilon]M^2(M + H + \gamma H)\left\{\frac{1}{K_r} + \left(\frac{1}{K_r} + 1\right)\sigma_f + Da_f M\left[\frac{1}{K_r} + 1 + 2\sigma_f\right] + Da_f^2 M^2\right\}}{m^2\left\{[1 + 2\sigma_f + Da_f M][\phi H + (1-\phi)\varepsilon(M + H)] + \beta\left[\frac{1}{K_r} + 2\sigma_f + Da_f M\right][\phi(M + \gamma H) + (1-\phi)\varepsilon\gamma H]\right\}}. \tag{34}$$

The minimum value of this expression, as the horizontal wavenumber m is varied, is the critical Rayleigh number (based on properties of the f-phase).

For perfectly conducting and rigid boundaries the boundary conditions are now

$$\Psi_f = \Psi_p = \frac{\partial \Psi_f}{\partial y} = \frac{\partial \Psi_p}{\partial y} = \Theta_f = \Theta_p = 0 \quad \text{at} \quad y = 0 \quad \text{and at} \quad y = 1. \tag{35}$$

A single-term Galerkin expansion gives the following approximate solution.

The solution of the system of Eqs. (27)–(31) is now

$$\Psi_f = A_1 T_1(y) \cos mx, \quad \Psi_p = A_2 T_2(y) \cos mx,$$
$$\Theta_f = A_3 T_3(y) \sin mx, \quad \Theta_p = A_4 T_4(y) \sin mx. \tag{36a, b, c, d}$$

An appropriate choice of the trial functions (so that the boundary conditions (35) are satisfied) is

$$T_1(y) = T_2(y) = y^2(1-y)^2, \quad T_3(y) = T_4(y) = y(1-y). \tag{37}$$

The standard procedure then leads to the eigenvalue equation

$$\begin{vmatrix} (1+\sigma_f)M_2 + Da_f M_3 & -\beta\sigma_f M_2 & \frac{9m\phi Ra_f}{2[\phi+(1-\phi)\varepsilon]} & \frac{9m(1-\phi)\varepsilon Ra_f}{2[\phi+(1-\phi)\varepsilon]} \\ -\sigma_f M_2 & \beta\left[\left(\frac{1}{K_r}+\sigma_f\right)M_2 + Da_f M_3\right] & \frac{9m\phi Ra_f}{2[\phi+(1-\phi)\varepsilon]} & \frac{9m(1-\phi)\varepsilon Ra_f}{2[\phi+(1-\phi)\varepsilon]} \\ \frac{3}{14}m & 0 & M_1+H & -H \\ 0 & \frac{3}{14}m & -\gamma H & M_1+\gamma H \end{vmatrix} = 0, \tag{38}$$

where $M_1 = 10 + m^2$, $M_2 = 12 + m^2$, $M_3 = 504 + 24m^2 + m^4$.

In turn, this leads to

$$Ra_f = \frac{28\beta[\phi + (1-\phi)\varepsilon]M_1(M_1 + H + \gamma H)\left\{\left[\frac{1}{K_r} + \left(\frac{1}{K_r} + 1\right)\sigma_f\right]M_2^2 + Da_f\left[\frac{1}{K_r} + 1 + 2\sigma_f\right]M_2 M_3 + Da_f^2 M_3^2\right\}}{27m^2\left\{[(1 + 2\sigma_f)M_2 + Da_f M_3][\phi H + (1-\phi)\varepsilon(M_1 + H)] + \beta\left[\left(\frac{1}{K_r} + 2\sigma_f\right)M_2 + Da_f M_3\right][\phi(M_1 + \gamma H) + (1-\phi)\varepsilon\gamma H]\right\}}. \tag{39}$$

The formula for the case of a regular porous medium is recovered as K_r, σ_f and ε tend to zero. The result is

$$Ra_f = \frac{M^2(M + H + \gamma H)(1 + Da_f M)}{m^2(M + \gamma H)}. \tag{34a}$$

This result is in agreement with Eq. (27) of Postelnicu and Rees [7].

It is interesting that for the case $K_r = 1$ a factor $1 + 2\sigma_f + Da_f\, M$ cancels and Eq. (34) reduces to

$$Ra_f = \frac{\beta M^2(M + H + \gamma H)(1 + Da_f M)}{m^2\{\phi H + (1-\phi)\varepsilon(M + H) + \beta[\phi(M + \gamma H) + ((1-\phi)\varepsilon\gamma H)]\}}, \tag{34b}$$

something which is independent of σ_f.

The corresponding result for a regular porous medium is

$$Ra_f = \frac{28M_1(M_1 + H + \gamma H)(M_2 + Da_f M_3)}{27m^2(M_1 + \gamma H)}. \tag{39a}$$

The special result for the case $K_r = 1$ is

$$Ra_f = \frac{28\beta[\phi + (1-\phi)\varepsilon]M_1(M_1 + H + \gamma H)(M_2 + Da_f M_3)}{27m^2\{\phi H + (1-\phi)\varepsilon(M_1 + H) + \beta[\phi(M_1 + \gamma H) + ((1-\phi)\varepsilon\gamma H)]\}}, \tag{39b}$$

For the classical Rayleigh–Bénard problem, the single-term Galerkin approximation gives a value 1750 for the critical Rayleigh number, compared with the exact value 1708, i.e., a value about 3% too large, well within the experimental error in a typical experiment.

The Rayleigh number based on effective properties of the porous medium, Ra, is related to Ra_f by

$$Ra = \frac{\gamma}{\gamma + 1} Ra_f. \tag{40}$$

Postelnicu and Rees [7] presented critical values of Ra_f but we have chosen to present critical values of Ra. We believe that this readily ensures a more meaningful comparison with results for the LTE limit (H tends to infinity).

3. Results and discussion

Since we have explicit analytical expressions available, our code for numerical results is simple, and we have checked our results against the precise values obtained by Postelnicu and Rees [7] for the regular medium case. Clearly we have a large number of parameters to investigate. We are guided by the fact that Postelnicu and Rees [7] investigated the variation of H (an internal heat exchange parameter) and γ (a modified thermal conductivity ratio). We have Ra_f as a function of m, Da_f, ϕ, ε, β, K_r, σ_f, γ, and H. Compared with the case of a regular porous medium investigated by Postelnicu and Rees [7] we have four new parameters, namely ε, β, K_r, and σ_f. We refer to these as the BDPM parameters, and it is the effect of these parameters that is our principal focus in this paper. Our results (unlike those for the regular medium) also depend on the value of ϕ. In this study we report results for the representative values $\phi = 0.4$ and $\varepsilon = 0.4$.

As a first step, we explored the space $K_r = [0.0001, 0.01, 1]$, $\sigma_f = [0.001, 1, 1000]$, $\beta = [0.1, 1, 10]$, and $Da_f = [0.001, 1]$. For this space we found that the critical wavenumber m_c did not vary much from the value 3. Accordingly we have reported values for the critical Rayleigh number only, and following Postelnicu and Rees [7] we have plotted this for

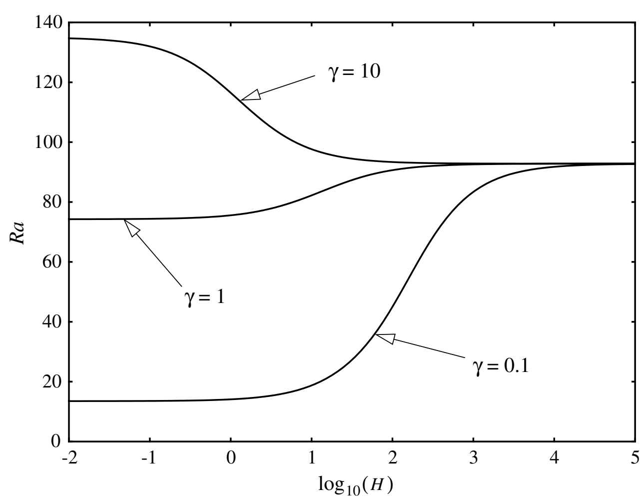

Fig. 2. Plots of the critical value of the Rayleigh number $Ra = \gamma Ra_f/(\gamma + 1)$ versus inter-phase heat transfer parameter H for various values of the modified thermal conductivity ratio γ. For each of Figs. 2–5 the parameter values are $\beta = 10$, $K_r = 0.0001$, $\phi = 0.4$, $\varepsilon = 0.4$. For this figure, $Da_f = 0.001$, $\sigma_f = 1$.

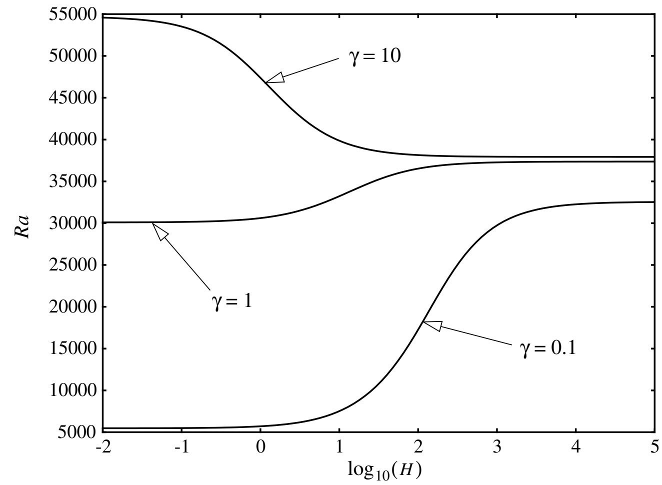

Fig. 3. Same as for Fig. 2, but now $\sigma_f = 1000$ instead of 1.

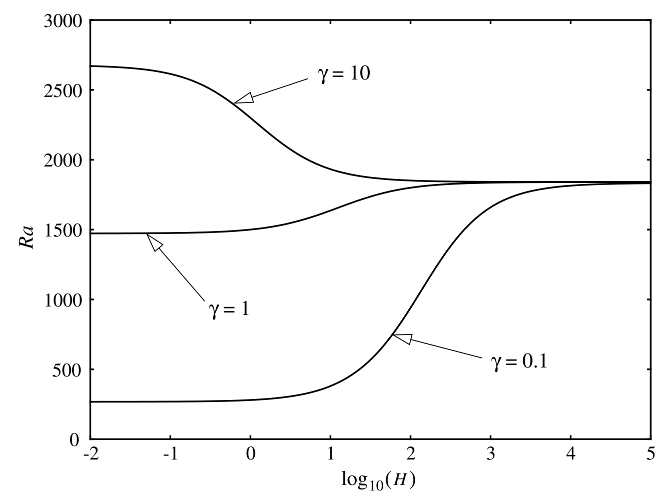

Fig. 4. Same as Fig. 2, but now $Da_f = 1$ instead of 0.001.

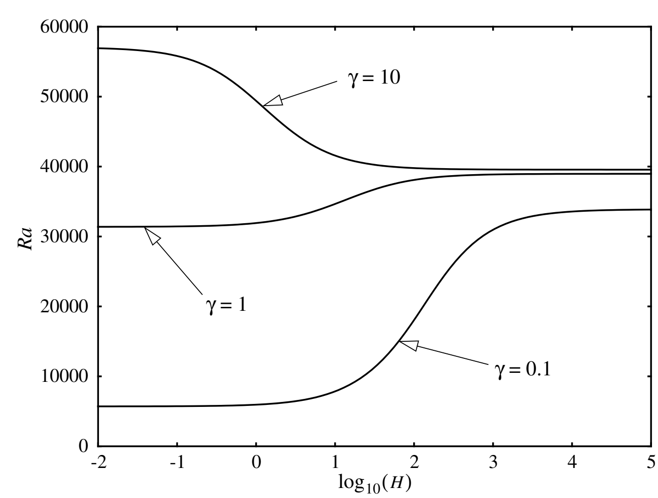

Fig. 5. Same as for Fig. 4, but now $\sigma_f = 1000$ instead of 1.

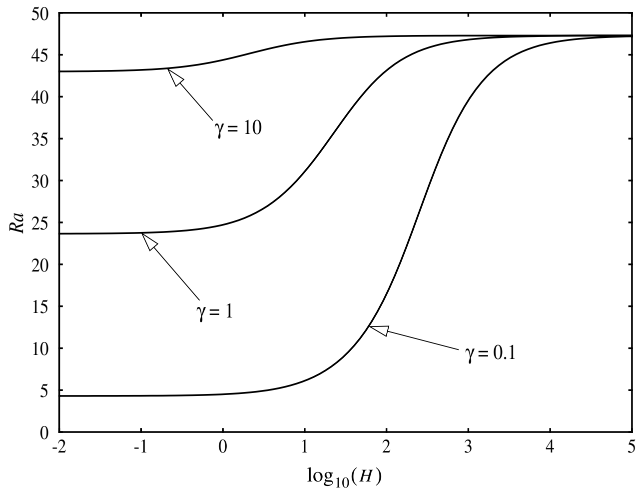

Fig. 6. Same as for Fig. 2 but for $\varepsilon = 0$, $K_r = 10^{-6}$, $\sigma_f = 0$. For this figure, $Da_f = 0.001$.

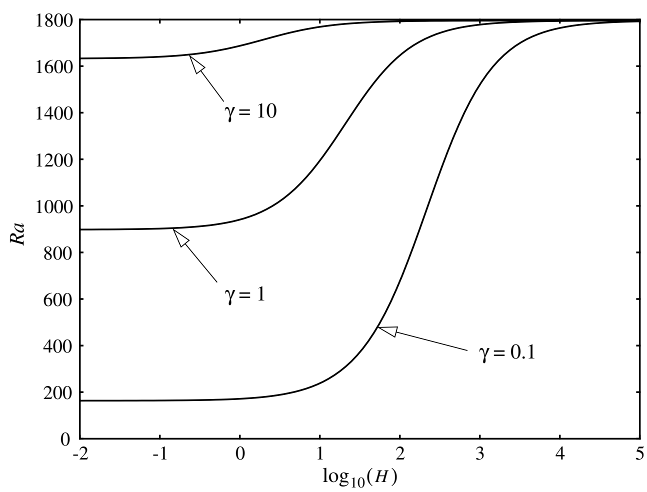

Fig. 7. Same as for Fig. 4, but for $\varepsilon = 0$, $K_r = 10^{-6}$, $\sigma_f = 0$. For this figure, $Da_f = 1$.

various values of γ and H. We also found that the general picture for the stress-free case was similar to that for the rigid body case, so we have reported numerical results for the latter case only. Also for this parameter range (54 sets of parameter values) we found the following. As we have already observed, when $K_r = 1$ the effect of a change in σ_f is zero. For smaller values of K_r, the effect of a change in σ_f from 0.001 to 1 is generally small, but the effect of a change from 1 to 1000 is significant, especially for very small K_r, and this effect becomes accentuated as β increases. We also observed that the interaction of the BDPM parameters and Da_f was not large.

With the above results of our preliminary investigation in mind, we have then concentrated on the case $\beta = 10$ and $K_r = 0.0001$, and effect of a change in σ_f from 1 to 1000, for each of a typical small Darcy number case ($Da_f = 0.001$) and a large Darcy number case ($Da_f = 1$). The results are displayed in Figs. 2–5. These plots show a substantial change in the critical value of Ra (as defined in Eq. (40)) from figure to figure, but the pattern of change with γ and H is much the same in each figure. These figures can be compared with Figs. 6 and 7, which correspond to the limiting case of a regular medium (very small values of K_r, ε, and σ_f). Comparing Fig. 2 with Fig. 6, and Fig. 4 with Fig. 7, we see that the main qualitative difference is the trend as H decreases (i.e., as one moves from LTE to LTNE) for the case of large γ. For the BDPM the critical Rayleigh number increases as H decreases, but for the regular porous medium the change is in the opposite direction.

We conclude this section with a general remark. We have introduced an inter-phase momentum transfer parameter σ_f on analogy with an inter-phase heat transfer parameter H. However, the analogy is not complete. The regular thermal situation (local thermal equilibrium) corresponds to the large H limit, but the regular hydrodynamic situation (a regular porous medium) corresponds to the small σ_f limit. In this pioneering study we have pushed the boundaries by reporting results for large values (as well as moderate values) of σ_f, but we recognize that these large values are probably not physically realistic. It appears that experimental results are currently lacking.

4. Conclusions

We have extended the classical Rayleigh–Bénard theory, for the onset of convection in a horizontal layer uniformly heated from below, for a regular porous medium (the Horton–Rogers–Lapwood problem), to the case of a bidisperse porous medium, using an extended Brinkman model. The extension involves the introduction of four additional dimensionless parameters, and we have investigated the effect of these. Two of the new parameters (an additional volume fraction and a permeability ratio) are essentially geometrical. The other two (a modified thermal capacity ratio and an inter-phase momentum transfer parameter) are of special interest.

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